Pergamon

Int. J. Solids Structures Vol. 31, No. 17, pp. 2367–2376, 1994

0020-7683/94 $7.00 + .00

0020-7683(94)E0019-R

DYNAMIC STABILITY OF NONLINEAR VISCOELASTIC PLATES

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(*Received* 13 *July* 1993; *in revised form* 20 *January* 1994)

Abstract—The dynamic stability analysis of isotropic plates made of a nonlinear viscoelastic material is performed within the concept of the Lyapunov exponents. The material behavior is modeled according to the Leaderman representation of nonlinear viscoelasticity. The influence of the various parameters involved on the possibility of instability to occur is investigated. It is also shown that in some cases the system is chaotic.

INTRODUCTION

The subject of the dynamic stability of structures subjected to in-plane loads is one of the most interesting problems in the field of structural vibration. When plates are considered, the phenomena can be observed, for example, in bridge dynamics or wingflutter (instability of aircraft in air flow). In the linear case, the behavior is governed by the Mathieu equation and the stability characterizations are given by the Strutt diagram. Instability here is in the sense that the amplitude of the response increases without bound. The problem was extensively investigated by Bolotin (1964) and further results were given, for example, by Evan-Iwanowski (1965, 1976) in a review paper and a monograph, respectively.

When the structure is made of a linear viscoelastic material, the problem becomes much more complicated since the equation of motion turns out to be an integro-differential one, rather than an ordinary differential equation as in the elastic case. To solve this problem, Matyash (1964) used the averaging method, while Stevens (1966), Szyskowski and Gluckner (1985) and Gluckner and Szyskowski (1987) analysed it by using the spring-dashpot representation. The dynamic stability of viscoelastic homogeneous plates investigated within the concept of the Lyapunov exponents was performed by Aboudi *et al.* (1990). This procedure was used also by Cederbaum *et al.* (1991) to investigate the dynamic stability of shear deformable viscoelastic laminated plates. In these two studies the Boltzmann superposition principle was incorporated, enabling the modeling of any linear viscoelastic material.

However, it is well known that many materials (e.g. polymers) are not linear and should be modeled nonlinearily in order to give an adequate description of their behavior. Smart and Williams (1972) made a comparison investigation about the response of polypropylene and polyvinylchloride, obtained by using three single integral representations of nonlinear viscoelasticity; the Leaderman model (1962), the Schapery model (1969) and the Bernstein–Kearsley–Zapas model (Bernstein *et al.*, 1963; Zapas and Craft, 1965). Their main conclusion was that the Leaderman model is the most useful representation where prediction and simplicity are concerned. In the present investigation we adopt this result and use the Leaderman model to derive the integro-differential equation of motion, which is nonlinear and with time-dependent coefficients.

The stability analysis of the nonlinear viscoelastic plate is based on the evaluation of the associate Lyapunov exponents. If one of the Lyapunov exponents is found to be positive then, according to Chetaev (1960), the unperturbed motion is unstable. Thus, in order to determine the condition of the plate, it suffices to compute the largest Lyapunov exponent only.

PROBLEM FORMULATION

The equation of motion of an isotropic plate subjected to in-plane loads ($N_{xy} = 0$) is [see e.g. Timoshenko (1963)]

$$M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + N_x W_{,xx} + N_y W_{,yy} + \rho h \ddot{W} = 0 \tag{1}$$

where N_x and N_y are in-plane loads in the x and y directions, respectively (see Fig. 1), W is the deflection in the transverse (z) direction, ρ is the material density and h is the plate thickness. The stress couples, M_{ij}, are given by

$$M_{ij} = -\int_{-h/2}^{h/2} z\sigma_{ij}\,\mathrm{d}z, \quad i, j = x, y \tag{2}$$

and σ_{ij} are the stress components. For nonlinear viscoelastic materials the stress–strain constitutive relations, as given by Leaderman (1962), are

$$\sigma(t) = Q(0)g[\varepsilon(t)] + \int_{0^+}^{t} \dot{Q}(t-\tau)g[\varepsilon(\tau)]\,\mathrm{d}\tau$$

where

$$g(\varepsilon) = \varepsilon + \beta\varepsilon^2 + \gamma\varepsilon^3 \tag{3}$$

which for small strain $g(\varepsilon) \to \varepsilon$. β and γ are constants. For the state of plane stress for isotropic plates

$$Q_{11}(t) = Q_{22}(t) = \frac{E(t)}{1-\nu(t)}$$

$$Q_{12}(t) = \nu(t)Q_{11}(t); \quad Q_{66}(t) = \tfrac{1}{2}(1-\nu(t))Q_{11}(t) \tag{4}$$

where $E(t)$ is a time-dependent relaxation function which at $t = 0$ denotes the initial Young modulus of the material, while $\nu(t)$ is the time-dependent Poisson ratio.

For a homogeneous thin plate, the strain–displacement relations are given by

$$\varepsilon_{xx} = -zW_{,xx}$$

$$\varepsilon_{yy} = -zW_{,yy}$$

$$\varepsilon_{xy} = -2zW_{,xy}. \tag{5}$$

Using the separation of variables method, the transverse displacement is written in the form

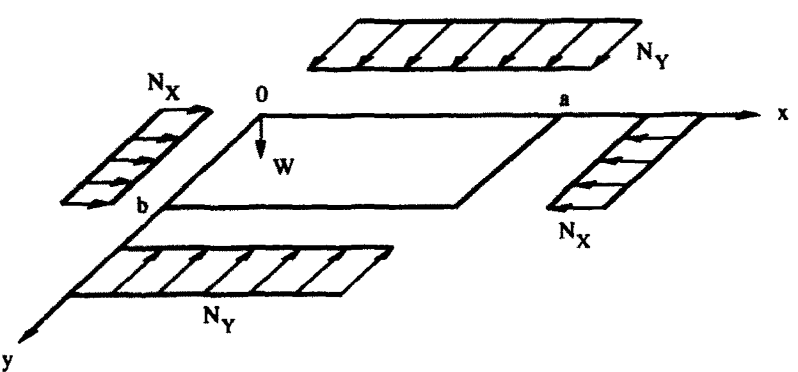

Fig. 1. Scheme of a plate subjected to in-plane loadings.

$$W(x,y,t) = \varphi(x,y)f(t) \tag{6}$$

so that by substituting eqns (5) and (6) into eqn (2), the stress couples are given by

$$\begin{aligned}
M_{xx} &= I_1\varphi_{,xx}\left[Q_{11}(0)f(t)+\int_{0^+}^{t}\dot{Q}_{11}(t-\tau)f(\tau)\,\mathrm{d}\tau\right] \\
&\quad + I_1\varphi_{,yy}\left[Q_{12}(0)f(t)+\int_{0^+}^{t}\dot{Q}_{12}(t-\tau)f(\tau)\,\mathrm{d}\tau\right] \\
&\quad + I_2(\varphi_{,xx})^3\left[Q_{11}(0)f^3(t)+\int_{0^+}^{t}\dot{Q}_{11}(t-\tau)f^3(\tau)\,\mathrm{d}\tau\right] \\
&\quad + I_2(\varphi_{,yy})^3\left[Q_{12}(0)f^3(t)+\int_{0^+}^{t}\dot{Q}_{12}(t-\tau)f^3(\tau)\,d\tau\right] \\
M_{yy} &= I_1\varphi_{,yy}\left[Q_{22}(0)f(t)+\int_{0^+}^{t}\dot{Q}_{22}(t-\tau)f(\tau)\,\mathrm{d}\tau\right] \\
&\quad + I_1\varphi_{,xx}\left[Q_{12}(0)f(t)+\int_{0^+}^{t}\dot{Q}_{12}(t-\tau)f(\tau)\,\mathrm{d}\tau\right] \\
&\quad + I_2(\varphi_{,yy})^3\left[Q_{22}(0)f^3(t)+\int_{0^+}^{t}\dot{Q}_{22}(t-\tau)f^3(\tau)\,\mathrm{d}\tau\right] \\
&\quad + I_2(\varphi_{,xx})^3\left[Q_{12}(0)f^3(t)+\int_{0^+}^{t}\dot{Q}_{12}(t-\tau)f^3(\tau)\,\mathrm{d}\tau\right] \\
M_{xy} &= 2I_1\varphi_{,xy}\left[Q_{66}(0)f(t)+\int_{0^+}^{t}\dot{Q}_{66}(t-\tau)f(\tau)\,\mathrm{d}\tau\right] \\
&\quad + 8I_3(\varphi_{,xy})^3\left[Q_{66}(0)f^3(t)+\int_{0^+}^{t}\dot{Q}_{66}(t-\tau)f^3(\tau)\,\mathrm{d}\tau\right]
\end{aligned} \tag{7}$$

where $I_1 = h^3/12$, $I_2 = \gamma_{xx}h^5/80$ and $I_3 = \gamma_{xy}h^5/80$.

The in-plane loadings, which contain constant and periodic terms, are given by

$$\begin{aligned}
N_x &= N_{xs}+N_{xd}\cos\theta t \\
N_y &= N_{ys}+N_{yd}\cos\theta t
\end{aligned} \tag{8}$$

where t is the time and θ is the load frequency. Using eqn (4) in eqn (7), together with eqn (8), eqn (1) is rewritten in the form

$$\begin{aligned}
&\nabla^4\varphi I_1\left[Q_{11}(0)f(t)+\int_{0^+}^{t}\dot{Q}_{11}(t-\tau)f(\tau)\,\mathrm{d}\tau\right]+\{I_2[(\varphi_{,xx})^3]_{,xx}+I_2[(\varphi_{,yy})^3]_{,yy}+I_2\nu[(\varphi_{,yy})^3]_{,xx} \\
&\quad + I_2\nu[(\varphi_{,xx})^3]_{,yy}+8I_3(1-\nu)[(\varphi_{,xy})^3]_{,xy}\}\left[Q_{11}(0)f^3(t)+\int_{0^+}^{t}\dot{Q}_{11}(t-\tau)f^3(\tau)\,\mathrm{d}\tau\right] \\
&\quad + (N_{xs}+N_{xd}\cos\theta t)\varphi_{,xx}f(t)+(N_{ys}+N_{yd}\cos\theta t)\varphi_{,yy}f(t)+\rho h\varphi\ddot{f}(t) = 0
\end{aligned} \tag{9}$$

where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}.$$

Since eqn (9) is a nonlinear partial differential equation with time-dependent coefficients, for which an exact solution is generally not available, we will use the Galerkin method for finding the unknown deflection $f(t)$ [see Bolotin (1964); Nayfeh and Mook (1979)]. In order to satisfy the boundary conditions of a simply-supported plate, the solution of the linear system [the linear system is given when substituting into eqn (9) $I_2 = I_3 = 0$] is given by

$$W(x, y, t) = f(t)\varphi(x, y) = f(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{10}$$

where a and b are the side lengths of the plate.

Following the Galerkin method, the following equation of motion is derived

$$\ddot{f}(t) + \Omega^2[1 - 2\eta\cos(\theta t)]f(t) + kf^3(t) = -\omega^2\int_{0^+}^{t}\dot{D}(t-\tau)f(\tau)\,\mathrm{d}\tau - k\int_{0^+}^{t}\dot{D}(t-\tau)f^3(\tau)\,\mathrm{d}\tau \tag{11}$$

where $a = b = \ell$ and

$$\Omega^2 = \omega^2\left[1 - \frac{N_{xs}+N_{ys}}{N}\right], \qquad \omega^2 = \frac{4\pi^4 I_1 Q_{11}(0)}{\ell^4 \rho h}$$

$$N = \frac{4\pi^2 I_1 Q_{11}(0)}{\ell^2}, \qquad D(t) = \frac{Q_{11}(t)}{Q_{11}(0)}$$

$$k = \frac{9\pi^8 Q_{11}(0)}{8\rho h \ell^8}[I_2(1+v) + 4I_3(1-v)], \qquad \eta = \frac{N_{xd}+N_{yd}}{2[N-(N_{xs}+N_{ys})]}. \tag{12}$$

Here, ω and Ω represent the natural frequency of lateral vibration of unloaded and loaded plates, respectively, N is the Euler critical load, η is the excitation parameter and k is the coefficient of nonlinearity.

Equation (11) is the nonlinear integro-differential equation which governs the motion of the nonlinear viscoelastic plate subjected to in-plane parametric loading.

METHOD OF SOLUTION

Interest here is in the stability of the unperturbed equilibrium of the nonlinear viscoelastic plate. To this end the integro-differential eqn (11) is investigated. For the treatment of nonlinear differential equations with time-dependent coefficients, Lyapunov introduced the concept of characteristic numbers, the sign of which determines whether or not the unperturbed motion is stable [see Hahn (1967)]. The negative values of these characteristic numbers are referred to as the Lyapunov exponents.

According to Lyapunov, if all the exponents are negative then the unperturbed motion is asymptotically stable. In addition, Chetaev (1960, 1961) showed that if one of the Lyapunov exponents is positive then the unperturbed motion is unstable. Thus, it suffices to compute the largest Lyapunov exponent in order to determine the stability of the unperturbed motion of the nonlinear viscoelastic plate in question. To derive the largest Lyapunov exponent of the system we use the following procedure [see e.g. Goldhirsch *et al.* (1987)]: consider the system of ordinary differential equations

$$\dot{X} = F(x, t);\tag{13}$$

for a given solution of eqn (13), $x(t)$, define the matrix

$$\mathbf{G}_{ij}[x(t)] = \left.\frac{\partial F_i(x)}{\partial x_j}\right|_{x=x(t)}.\tag{14}$$

The largest Lyapunov exponent is then determined by solving the equations

$$\dot{y} = \mathbf{G}y\tag{15}$$

and performing the following steps:

1. for the first time interval, Δt, solve eqn (15) by considering initial conditions, $y(0)$, normalized such that $\|y(0)\| = 1$, where $\|\cdot\|$ is the Euclidean norm;
2. compute $\mu_1 = \ln\|y(\Delta t)\|$;
3. let $z(\Delta t) = y(\Delta t)/\|y(\Delta t)\|$;
4. for the second time interval, $2\Delta t$, solve eqn (15) with $z(\Delta t)$ as the initial condition [**G** has to be changed according to eqn (14)] and determine $\mu_2 = \ln\|y(2\Delta t)\|$;
5. repeat the process for n iterations.

One defines then

$$\lambda_1 = \frac{\sum_{m=1}^{n} \mu_m}{n\Delta t}\tag{16}$$

which, for $n \to \infty$, is the largest Lyapunov exponent.

In order to compute λ_1, the governing eqn (11) must be reduced to a system of first-order equations of the form of eqn (15). Consider a linear solid material for which the relaxation function is given by

$$E(t) = a + b\,\mathrm{e}^{-\alpha t}\tag{17}$$

where a, b and α are appropriate parameters. Thus, for material with time-independent Poisson ratio

$$Q_{11}(t) = E(t)/(1-v^2) = A + B\,\mathrm{e}^{-\alpha t}\tag{18}$$

so that

$$D(t) = \frac{Q_{11}(t)}{Q_{11}(0)} = \frac{A + B\,\mathrm{e}^{-\alpha t}}{A+B}\tag{19}$$

and after differentiating, can be written as

$$\dot{D}(t-\tau) = -\psi_1(t)\cdot\psi_2(\tau)\tag{20}$$

where

$$\psi_1(t) = \frac{B}{A+B}\mathrm{e}^{-\alpha t} \quad \text{and} \quad \psi_2(\tau) = \alpha\,\mathrm{e}^{-\alpha\tau}.$$

Substituting eqn (20) into eqn (11), we obtain

$$\frac{\ddot{f}(t)}{\psi_1}+\frac{\Omega^2}{\psi_1}[1-2\eta\cos\theta t]f(t)+\frac{k}{\psi_1}f^3(t)=\int_{0^+}^{t}\psi_2(\tau)[\omega^2 f(\tau)+kf^3(\tau)]\,d\tau \tag{21}$$

which, after differentiating via Leibnitz's rule, the following ordinary differential equation is derived

$$\dddot{f}(t)-\frac{\dot{\psi}_1(t)}{\psi_1(t)}\ddot{f}(t)+[\Omega^2(1-2\eta\cos\theta t)+3kf^2(t)]\dot{f}(t)$$
$$+\left[-\Omega^2(1-2\eta\cos\theta t)\frac{\dot{\psi}_1(t)}{\psi_1(t)}+\Omega^2 2\eta\theta\sin(\theta t)\right.$$
$$\left.-\omega^2\psi_1(t)\psi_2(t)\right]f(t)-k\left[\psi_2(t)\psi_1(t)+\frac{\dot{\psi}_1(t)}{\psi_1(t)}\right]f^3(t)=0. \tag{22}$$

Finally, by using eqn (20), eqn (22) is written as

$$\dddot{f}(t)+\alpha\ddot{f}(t)+[\Omega^2(1-2\eta\cos\theta t)+3kf^2(t)]\dot{f}(t)$$
$$+\left[\alpha\Omega^2(1-2\eta\cos\theta t)+2\eta\theta\Omega^2\sin\theta t-\frac{\alpha B\omega^2}{A+B}\right]f(t)+\frac{k\alpha A}{A+B}f^3(t)=0. \tag{23}$$

Equation (23) can be written in the form of eqn (13). In addition, the various elements of the matrix **G** are given by

$$\mathbf{G}_{11}=\mathbf{G}_{13}=\mathbf{G}_{21}=\mathbf{G}_{22}=0$$
$$\mathbf{G}_{12}=\mathbf{G}_{23}=1$$
$$\mathbf{G}_{31}=-\alpha\Omega^2(1-2\eta\cos\theta t)-2\eta\theta\Omega^2\sin(\theta t)+\frac{\alpha}{A+B}(B\omega^2-3kAf^2)-6kf\dot{f}$$
$$\mathbf{G}_{32}=-\Omega^2(1-2\eta\cos\theta t)-3kf^2;\quad \mathbf{G}_{33}=-\alpha. \tag{24}$$

NUMERICAL RESULTS AND DISCUSSION

In this section the stability of eqn (23) is analysed with respect to the various parameters involved. The solution of this equation and of eqn (15) is obtained within the Runge–Kutta method (Matlab, 1991). First, it is recognized that for the case where $\alpha = k = 0$, one obtains the well-known Mathieu equation, which was extensively investigated, e.g. by McLachlan (1964). When $k = 0$ and $\alpha \neq 0$ we have

$$\ddot{f}+\Omega^2[1-2\eta\cos\theta t]f=-\omega^2\int_{0^+}^{t}\dot{D}(t-\tau)f(\tau)\,d\tau \tag{25}$$

which describes the motion of a linear viscoelastic structure. The stability of this equation was investigated by Aboudi *et al.* (1990) by using the concept of Lyapunov exponents. Later on, Cederbaum and Mond (1992) and Cederbaum (1992) investigated this equation analytically, and obtained the expression for the critical (minimum) value of the excitation parameter, η_c, at which instability may occur. For the case of the standard linear solid model it is

$$\eta_c=\frac{2}{\theta}|\dot{D}(0)|=\frac{2}{\theta}\frac{\alpha B}{A+B} \tag{26}$$

and it will be used later on. For the case where $\alpha = 0$ and $k \neq 0$, one obtains

$$\ddot{f}+\Omega^2[1-2\eta\cos\theta t]f+kf^3 = 0 \tag{27}$$

representing a nonlinear version of the Mathieu equation, and which was examined, e.g. by Bolotin (1964).

In the following, we consider the general case where $\alpha \neq 0$ and $k \neq 0$. The numerical results were obtained by using $A = 0.1$ and $B = 0.9$, and where $N_{xs} = N_{ys} = 0$, $\Omega = \omega = 1$ and $\theta = 2\omega$.

Figure 2 shows the response, $f(t)$, as well as the largest Lyapunov exponent, λ_1, derived for the case where $\alpha = 0.01$, $k = 0.01$ and η is equal to (a) 0.004, (b) 0.009 ($= \eta_c$) and (c)

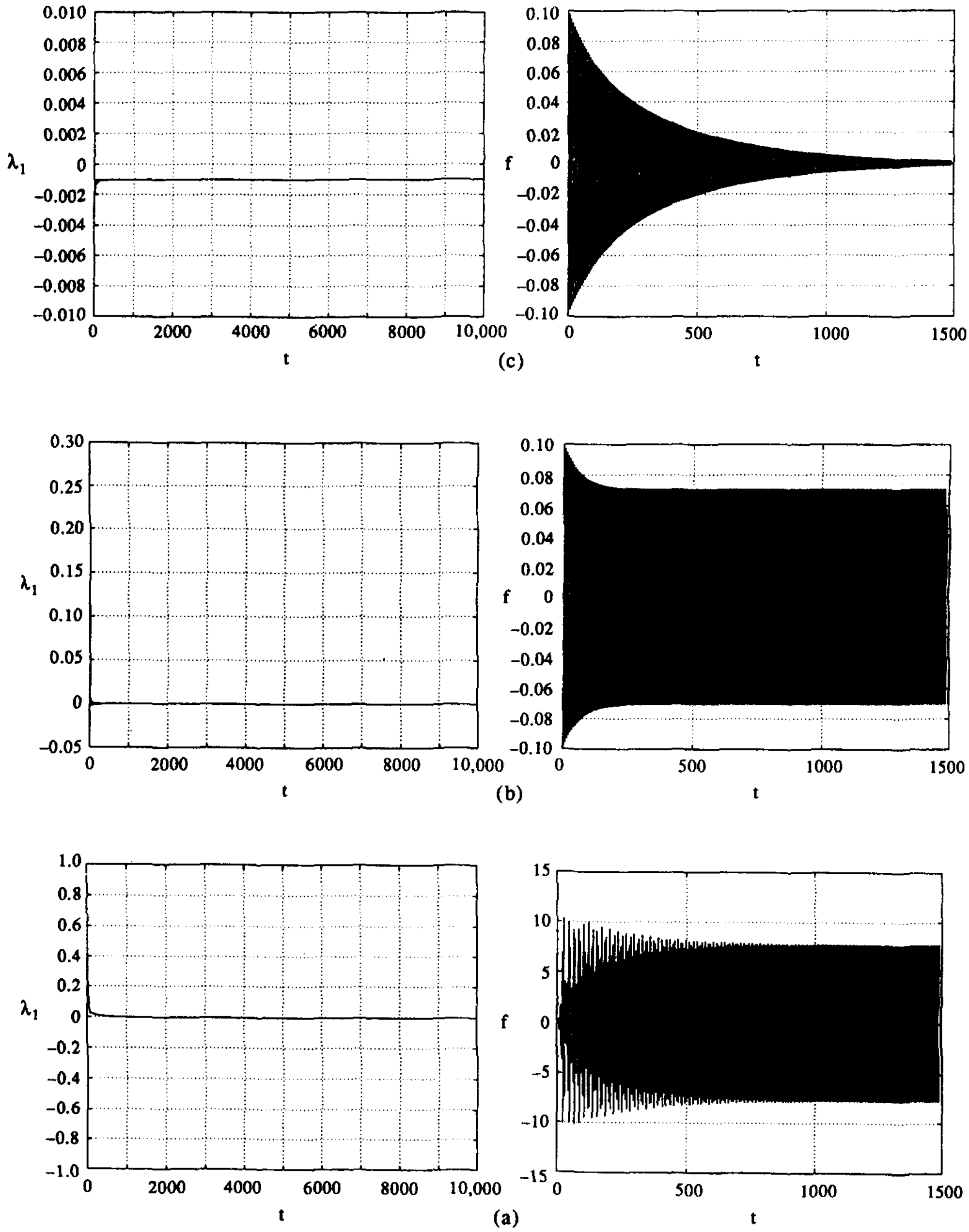

Fig. 2. The response, f, and the largest Lyapunov exponent, λ_1, for $\alpha = 0.01$, $k = 0.01$ and (a) $\eta = 0.004$, (b) $\eta = 0.009$, (c) $\eta = 0.5$.

0.5. In Fig. 2(a) the system is asymptotically stable, that is λ_1 is negative and the response is approaching zero. In Figs 2(b) and (c) the system is stable with limit cycle and $\lambda_1 \to 0$. Yet, in Fig. 2(c) the amplitude is much larger than that in Fig. 2(b) [when $\eta > \eta_c$, the amplitude can be approximated by $A = 1/\sqrt{k}$, see e.g. Bolotin (1964)].

In Fig. 3, $k = 0.01$, $\eta = 0.5$ and the following cases for α are considered; (a) 0, (b) 0.000001 and (c) 0.0001. In Figs 3(a) and (b), λ_1 is positive, indicating instability. For relatively large α [case (c)] $\lambda_1 \to 0$ and the system is stable.

The response and the largest Lyapunov exponent shown in Fig. 4 are for the cases where $\alpha = 0.000001$, $\eta = 0.5$ and (a) $k = 0$, (b) $k = 0.00001$ and (c) $k = 1$. Figure 4(a) represents a linear viscoelastic case with $\eta > \eta_c$ and thus the system is unstable with positive λ_1 and amplitude which grow exponentially. In the nonlinear case, Fig. 4(b), the system is also unstable (positive Lyapunov exponent), but with finite amplitude. In Fig. 4(c) $\lambda_1 \to 0$ so that the system is stable (with relatively small amplitude).

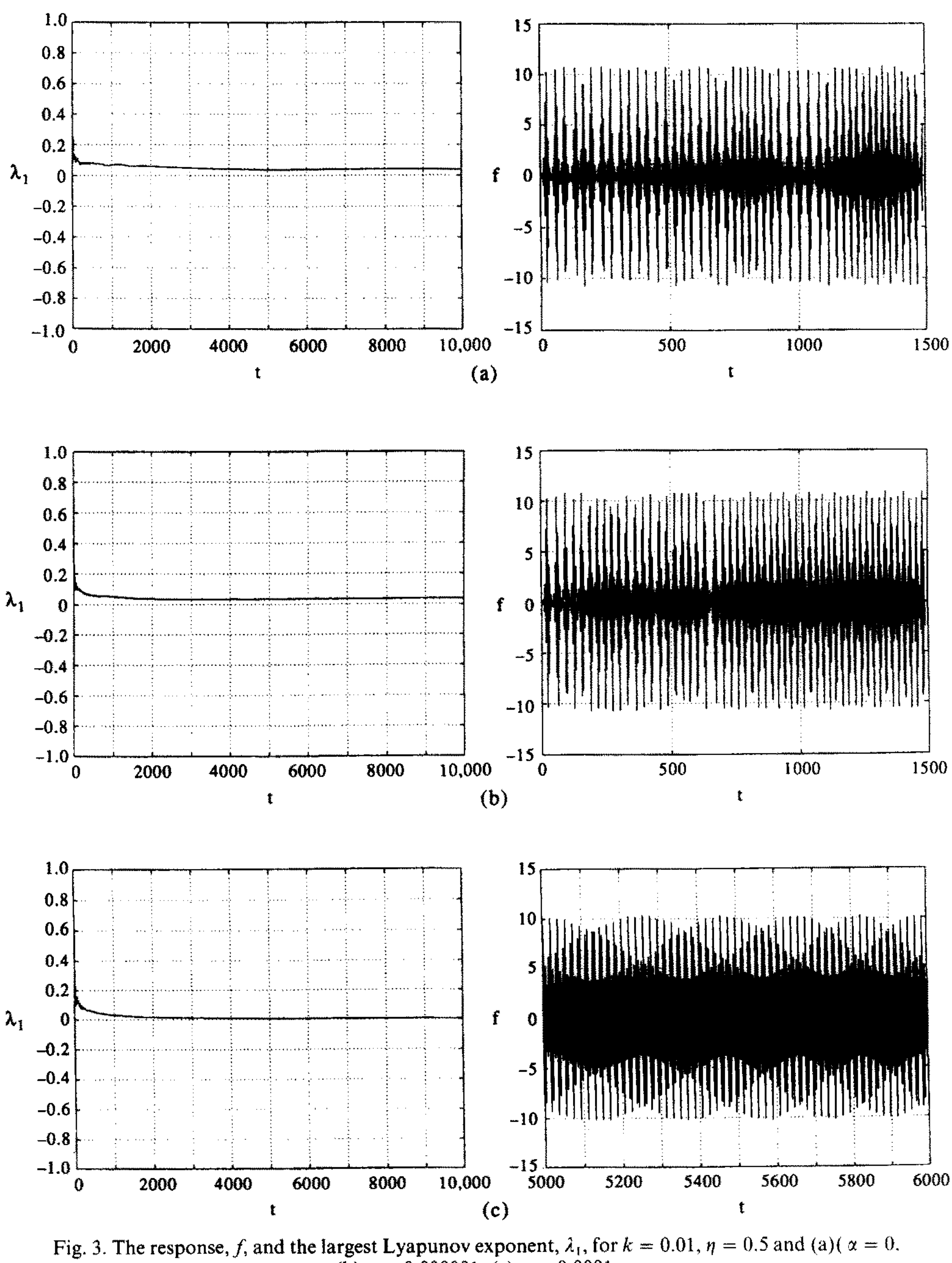

Fig. 3. The response, f, and the largest Lyapunov exponent, λ_1, for $k = 0.01$, $\eta = 0.5$ and (a)($\alpha = 0$, (b) $\alpha = 0.000001$, (c) $\alpha = 0.0001$.

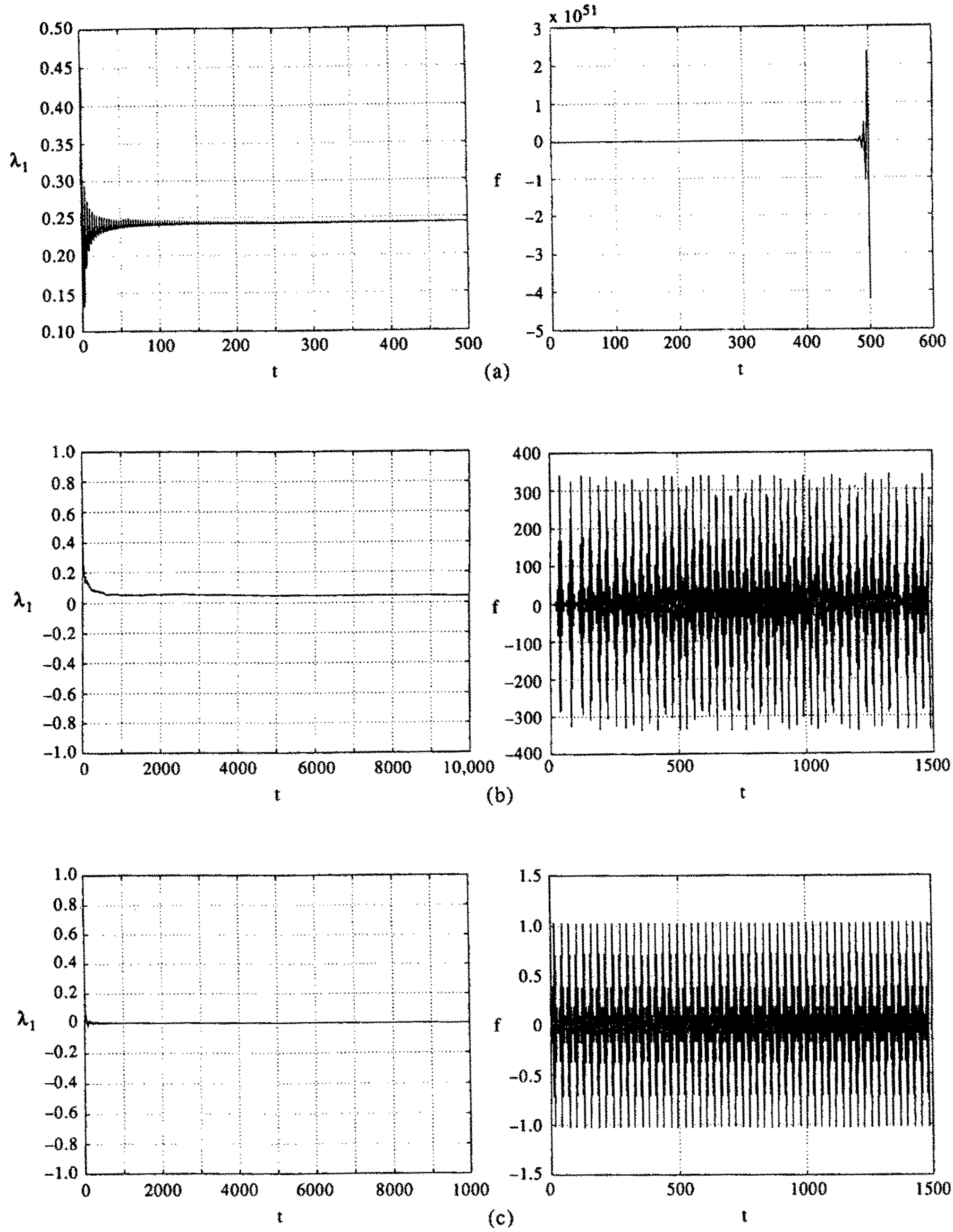

Fig. 4. The response, f, and the largest Lyapunov exponent for $\alpha = 0.000001$, $\eta = 0.5$ and (a) $k = 0$, (b) $k = 0.00001$, (c) $k = 1$.

From the above we may conclude the following.

1. Due to the nonlinear viscoelasticity, the response remains bounded even at instability (contrary to the case of linear viscoelastic material). Moreover, high nonlinearity stabilizes the system, as compared with the unstable case with nonlinearity [see Fig. 4(c)].
2. The material coefficient, α, has a great influence on the system in the sense that an unstable system may become stable at large values of α [see Fig. 3(c)]. The above is correct at $\eta > \eta_c$. But α is one of the parameters by which η_c is determined by [see eqn (28)], in a way that at large α, η_c is increased so that α stabilizes the system in this respect too.
3. At $\eta < \eta_c$, the system is asymptotically stable regardless of the values of α and k.

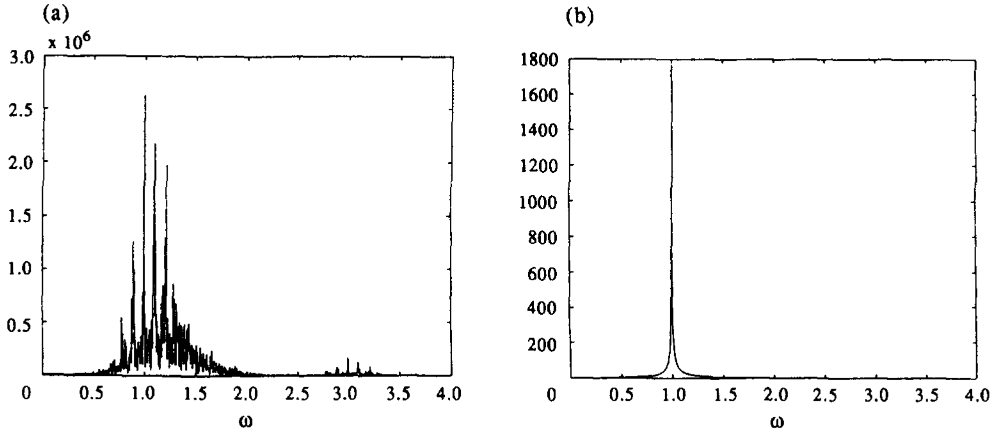

Fig. 5. The Fourier power spectrum of (a) the case of Fig. 4(b), (b) the case of Fig. 2(b).

Finally, it is noted that the Lyapunov exponents are served also as a powerful tool in the study of chaotic motion, and actually the existence of at least one positive Lyapunov exponent indicates a chaotic state [see e.g. Moon (1987); Wolf *et al.* (1985); Goldhirsch (1987)]. This state may also be realized if the response has a broad spectrum of frequencies. Figure 5(a) exhibits the Fourier power spectrum of the instability case given in Fig. 4(b), while Fig. 5(b) shows the same but for the stable case shown in Fig. 2(b). Thus, we believe that more attention should be given to the chaotic state possible in this problem, and this is the aim of future work.

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